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Space-eigenvalue problems in the kinetic theory of gases

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The existence of elementary, exponential solutions of the linear Boltzmann equation for gases is examined. Using the hard-sphere model of scattering, it is found that, in problems involving velocity perturbations, there are no discrete non-zero eigenvalues. Thus the relaxation to the asymptotic distribution is non-exponential and is described by the continuum eigenfunctions. For temperature perturbations, however, we find two non-zero discrete eigenvalues whose values are $\pm 0{\cdot}975$ in units of the minimum scattering cross-section. Relaxation to the asymptotic distribution is therefore exponential, although still very rapid.

The conclusions stated above are based upon a truncation of the scattering kernel and a subsequent numerical solution of the resulting integral equations.

1. Introduction

The perturbations introduced into the molecular distribution function of a gas due to temperature and velocity gradients were studied many years ago by Maxwell and by Boltzmann, both of whom obtained the well-known proportionality relationships between heat flow and temperature gradient, and pressure tensor and velocity gradient. These calculations were made more precise by Chapman (1917) and Enskog (1911), who were able to devise a method for solving the non-linear Boltzmann equation in an infinite expanse of gas sustaining small velocity and temperature gradients. Unfortunately, such solutions are of an asymptotic nature and throw very little light on the behaviour of a gas in the neighbourhood of a wall or solid boundary. However, through the work of Kramers (1949), Wang-Chang & Uhlenbeck (1953, 1954, 1956), Mott-Smith (1954) and Welander (1954) it has been shown that the presence of a wall modifies the distribution function, obtained by Chapman & Enskog, up to several mean free paths from the boundary. This surface region is called the Knudsen layer, and the manner and rate at which the distribution function changes within it is of considerable interest.

The recent advances in kinetic theory have been concerned with a greater understanding of the Knudsen layer: the work of Cercignani and his co-workers being worthy of particular comment in this respect (Cercignani 1962). Cercignani's approach has been to use simplified models of the molecular scattering process, thus enabling exact, analytic solutions of the linearized Boltzmann equation to be obtained. Complementary work by Loyalka & Ferziger (1968) has employed the variational method with more realistic scattering models. Irrespective of the method of solution of the Boltzmann equation, the general conclusion reached by all workers is that the macroscopic quantities $Q_i(x)$ (where Q_i refers to density, velocity, heat flux, etc.) can always be written in the following form:

$$Q_i(x) = Q_i^{\text{asy}}(x) + Q_i^{\text{trans}}(x), \tag{1}$$

with x the distance from the wall.

Apart from possibly a constant factor, $Q_i^{\text{asy}}(x)$ is the value obtained from the Chapman–Enskog approach, and it usually dominates the solution a few mean free paths from the surface at $x = 0$. $Q_i^{\text{trans}}(x)$ describes the Knudsen layer, and is generally small a few mean free paths from $x = 0$.

The purpose of the present paper is to examine the analytical nature of Q_i^{trans} in more detail than has previously been done. We shall accomplish this by studying the existence of elementary solutions of the transport equation in the form

$$h(\mathbf{c}, x) = \tilde{h}_\nu(\mathbf{c})\, e^{-\nu x}. \tag{2}$$

The allowed values of ν will be studied for velocity and temperature variations using the hard-sphere model of molecular scattering.

2. Basic theory

If the solution to the non-linear Boltzmann equation is written as

$$f(\mathbf{c}, x) = f_0(\mathbf{c})\{1 + h(\mathbf{c}, x)\}, \tag{3}$$

where $f_0(\mathbf{c})$ is the equilibrium Maxwell–Boltzmann distribution at temperature T_0, and h is a small perturbation, then the equation may be linearized by neglecting terms of $O(h^2)$ and higher. The result can be written in the following form (Williams 1969):

$$\begin{aligned}\left[c\mu \frac{\partial}{\partial x} + V(c)\right] & h(c, \mu, \chi, x) \\ &= \int_0^\infty dc'\, c'^2 e^{-c'^2} \int_{-1}^{1} d\mu' \int_0^{2\pi} d\chi'\, K(c, c', \mu, \mu', \chi, \chi')\, h(c', \mu', \chi', x),\end{aligned} \tag{4}$$

where we have assumed plane symmetry. $V(c)$ is the collision rate, and $K(\ldots)$ is the scattering kernel for molecular collisions. c is a reduced velocity defined by $c = v(m/2kT_0)^{\frac{1}{2}}$. The angular co-ordinates μ and χ are defined by Williams (1969, figure 1).

Equation (4) has associated with it certain boundary conditions on the surfaces at $x = \pm a$ (say). These boundary conditions are determined by the nature of the particle wall interaction and can be written in the general form,

$$c_x f(\mathbf{c}, -a) = -\int_{c_x' < 0} d\mathbf{c}'\, c_x'\, \Gamma(\mathbf{c}, \mathbf{c}') f(\mathbf{c}', -a), \tag{5}$$

for $c_x > 0$. There is a similar condition at $x = +a$ with the signs of c_x and c_x' reversed. $\Gamma(\mathbf{c}, \mathbf{c}')$ is the wall-particle scattering kernel. We shall not comment further on these boundary conditions, since they are not used explicitly in the present work.

Now, if we assume that the problem of interest involves velocity and temperature changes, we shall require an equation for the flow velocity parallel to the surface, i.e.

$$q(x) = \left(\frac{2kT_0}{m}\right)^{\frac{1}{2}} \frac{\int d\mathbf{c}\, c_z f(\mathbf{c},x)}{\int d\mathbf{c} f(\mathbf{c},x)} \tag{6}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty dc\, c^3\, e^{-c^2} \int_{-1}^{1} d\mu (1-\mu^2)^{\frac{1}{2}} g(c,\mu,x), \tag{7}$$

where
$$g(c,\mu,x) = \frac{1}{\pi} \int_0^{2\pi} d\chi \cos\chi h(c,\mu,\chi,x).$$

Also, we require the temperature $T(x)$, namely

$$\tfrac{3}{2}kT(x) = \tfrac{1}{2}m \frac{\int d\mathbf{c}\, v^2 f(\mathbf{c},x)}{\int d\mathbf{c} f(\mathbf{c},x)}, \tag{8}$$

or
$$T(x) = T_0\left[1 - \frac{4}{3\sqrt{\pi}} \int_0^\infty dc\, c^2(\tfrac{3}{2}-c^2)\, e^{-c^2} \int_{-1}^{1} d\mu\, \phi(c,\mu,x)\right], \tag{9}$$

where
$$\phi(c,\mu,x) = \frac{1}{2\pi} \int_0^{2\pi} d\chi\, h(c,\mu,\chi,x).$$

Equations for g and ϕ may be obtained directly from (4). Thus, after using the expansion of $K(\ldots)$ in Legendre polynomials (Williams 1969), we find

$$\left[c\mu \frac{\partial}{\partial x} + V(c)\right] g(c,\mu,x)$$
$$= \sum_{l=1}^{\infty} \frac{2l+1}{2l(l+1)} P_l^{(1)}(\mu) \int_0^\infty dc'\, c'^2\, e^{-c'^2} K_l(c,c') \int_{-1}^{1} d\mu'\, P_l^{(1)}(\mu')\, g(c',\mu',x), \tag{10}$$

and

$$\left[c\mu \frac{\partial}{\partial x} + V(c)\right] \phi(c,\mu,x)$$
$$= \sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(\mu) \int_0^\infty dc'\, c'^2\, e^{-c'^2} K_l(c,c') \int_{-1}^{1} d\mu'\, P_l(\mu')\, \phi(c',\mu',x). \tag{11}$$

These are the basic equations for study.

3. The eigenvalue problems

As explained in the introduction, we shall seek solutions to (10) and (11) in the following forms:

$$g(c,\mu,x) = g_\nu(c,\mu)\, e^{-\nu x}, \tag{12}$$

and
$$\phi(c,\mu,x) = \phi_\kappa(c,\mu)\, e^{-\kappa x}. \tag{13}$$

The allowed values of ν and κ will now be studied.

Inserting (12) and (13) into (10) and (11), respectively, leads to the following eigenvalue problems:

$$[V(c) - \nu c\mu]\, g_\nu(c,\mu) = \sum_{l=1}^{\infty} \hat{O}_l g_\nu(c,\mu), \tag{14}$$

and
$$[V(c) - \kappa c\mu]\, \phi_\kappa(c,\mu) = \sum_{l=0}^{\infty} \hat{Q}_l \phi_\kappa(c,\mu), \tag{15}$$

where we have abbreviated the right-hand sides of (10) and (11) by the operators $\hat{O}_l$ and $\hat{Q}_l$, respectively.

In order to calculate the eigenvalues ν and κ we divide by the quantities in square brackets on the left-hand sides of (14) and (15), multiply by $P_m^{(1)}(\mu)$ and $P_m(\mu)$, respectively, and integrate over $\mu(-1, 1)$. We then obtain

$$g_m(c) = \sum_{l=1}^{\infty} B_{lm}(c,\nu) \int_0^{\infty} dc'\, c'^2 e^{-c'^2} K_l(c,c')\, g_l(c'), \tag{16}$$

and

$$\phi_m(c) = \sum_{l=0}^{\infty} A_{lm}(c,\kappa) \int_0^{\infty} dc'\, c'^2 e^{-c'^2} K_l(c,c')\, \phi_l(c'), \tag{17}$$

where

$$g_l(c) = \int_{-1}^{1} d\mu'\, P_l^{(1)}(\mu')\, g_\nu(c,\mu'),$$

and

$$\phi_l(c) = \int_{-1}^{1} d\mu'\, P_l(\mu')\, \phi_\kappa(c,\mu').$$

Also, we have defined

$$B_{lm}(c,\nu) = \frac{2l+1}{2l(l+1)} \int_{-1}^{1} \frac{P_l^{(1)}(\mu)\, P_m^{(1)}(\mu)\, d\mu}{V(c)-\nu c\mu}, \tag{18}$$

and

$$A_{lm}(c,\kappa) = \frac{2l+1}{2} \int_{-1}^{1} \frac{P_l(\mu)\, P_m(\mu)\, d\mu}{V(c)-\kappa c\mu}. \tag{19}$$

We have, therefore, a set of coupled integral equations for the eigenvalues ν and κ.

In order to proceed further it is necessary to know the behaviour of $V(c)$ with c. If we choose the hard-sphere model of scattering, then

$$V(c) = \frac{1}{\sqrt{\pi}} e^{-c^2} + \left(c + \frac{1}{2c}\right) \operatorname{erf}(c). \tag{20}$$

We should note that, in arriving at (16) and (17), it was implicitly assumed that

$$[V(c) - \nu c\mu] \neq 0, \tag{21}$$

$$[V(c) - \kappa c\mu] \neq 0; \tag{22}$$

thus, in view of the nature of $V(c)$ (namely $V(c) \to c$ as $c \to \infty$, and $V(c) \to 2/\sqrt{\pi}$ as $c \to 0$), it is clear that ν and κ, as determined by (16) and (17), are restricted as follows:

$$-1 < \operatorname{Re}(\nu) < 1, \tag{23}$$

$$-1 < \operatorname{Re}(\kappa) < 1. \tag{24}$$

In fact, the limits are really $-\Sigma_{\min}$ and $\Sigma_{\min}$, since we have chosen our units such that the minimum value of the scattering cross-section $V(c)/c = \Sigma(c)$ is equal to unity. Thus the discrete eigenvalues ν and κ have limit points at ± 1. For values of ν and κ exceeding these limit points the associated eigenfunction becomes singular. Indeed, in the range of ν and κ lying between 1 and ∞ and -1 and $-\infty$, g_ν takes the form,

$$g_\nu(c,\mu) = P.\frac{1}{V(c)-\nu c\mu} \sum_{l=1}^{\infty} \hat{O}_l g_\nu(c,\mu) + \lambda(c,\nu)\, \delta(V(c)-\nu c\mu), \tag{25}$$

where the symbol $P.$ indicates that principal value integrals are involved in integration over g_ν, and $\delta(\ldots)$ is the Dirac delta function. Similarly for ϕ_κ. Such functions are known as singular eigenfunctions and, together with any discrete eigenfunctions arising from solutions of (16) and (17), can be shown to form a complete set (Koppel 1963; Grad 1964).

The purpose of the present work is to investigate the existence of the discrete eigenvalues lying in the range $(-1, 1)$. Bearing in mind the conservation laws, it is not difficult to see that the components of the kernel $K(\mathbf{c}, \mathbf{c}')$ obey the following relationships, namely

$$V(c) = \int_0^\infty dc'\, c'^2 e^{-c'^2} K_0(c, c'), \tag{26}$$

$$c^2 V(c) = \int_0^\infty dc'\, c'^4 e^{-c'^2} K_0(c, c'), \tag{27}$$

$$c V(c) = \int_0^\infty dc'\, c'^3 e^{-c'^2} K_1(c, c'), \tag{28}$$

where K_0 and K_1 are given explicitly for hard-spheres in the appendix.

With the use of (28), it is easy to see that (16) has the solution

$$g_l(c) = c\delta_{l,1}, \tag{29}$$

corresponding to the double eigenvalue $\nu = \pm 0$. This corresponds to momentum conservation and is related to the asymptotic Chapman–Enskog solution.

With (26) and (27), we note that (17) has solutions

$$\phi_l(c) = \delta_{l,0}, \tag{30}$$

and

$$\phi_l(c) = c^2\delta_{l,0}, \tag{31}$$

each corresponding to a double eigenvalue at $\kappa = \pm 0$.

Equations (30) and (31) also correspond to the Chapman–Enskog part of the solution.

The question of most interest is that of the existence of any non-zero eigenvalues in the range $(-1, 1)$, for these will govern the relaxation process from the boundary plane. Clearly, it is not possible to solve (16) and (17) in complete generality. However, it is clear that any non-zero eigenvalues are real and occur in the form $\pm \nu$. Our procedure is to retain only the first term in the expansions of the scattering kernel. Thus, (16) becomes

$$g_1(c) = B_{11}(c, \nu) \int_0^\infty dc'\, c'^2 e^{-c'^2} K_1(c, c')\, g_1(c'), \tag{32}$$

and (17) becomes

$$\phi_0(c) = A_{00}(c, \kappa) \int_0^\infty dc'\, c'^2 e^{-c'^2} K_0(c, c')\, \phi_0(c'), \tag{33}$$

where A_{00} and B_{11} are given explicitly by

$$A_{00} = \frac{1}{2c\kappa} \log\left(\frac{\Sigma(c) + \kappa}{\Sigma(c) - \kappa}\right), \tag{34}$$

and

$$B_{11} = \frac{3\Sigma(c)}{2c\nu^2} + \frac{3}{4c\nu}\left(1 - \frac{\Sigma^2(c)}{\nu^2}\right) \log\left(\frac{\Sigma(c) + \nu}{\Sigma(c) - \nu}\right), \tag{35}$$

where we have set $V(c) = c\Sigma(c)$.

The probable error involved in this truncation procedure will be discussed in a later section.

4. Numerical procedure

Equations (32) and (33) are implicit eigenvalue problems for ν and κ. We find it more convenient, therefore, to introduce a fictitious eigenvalue, $\rho(\nu)$, and to write (32) as

$$\rho(\nu)\, g_1(c) = B_{11}(c,\nu)\int_0^\infty dc'\, c'^2 e^{-c'^2} K_1(c,c')\, g_1(c'), \tag{36}$$

and similarly with $\rho(\kappa)$ for (33). Equation (36) is now in canonical form, and when $\rho = 1$ it corresponds to the problem of interest. Our procedure therefore is to seek $\rho(\nu)$ as a function of ν and obtain the desired value of ν as the root of $\rho(\nu) = 1$ or $\rho(\kappa) = 1$ (Wood 1966). First, however, let us symmetrize (36), whence we find

$$\rho(\nu)\, \Psi_\nu(c) = \int_0^\infty dc'\, H_\nu(c,c')\, \Psi_\nu(c'), \tag{37}$$

where
$$\Psi_\nu(c) = \frac{c\, e^{-\frac{1}{2}c^2} g_1(c)}{[B_{11}(c,\nu)]^{\frac{1}{2}}}, \tag{38}$$

and
$$H_\nu(c,c') = cc'\, e^{-\frac{1}{2}(c^2+c'^2)} K_1(c,c')\, [B_{11}(c,\nu)\, B_{11}(c',\nu)]^{\frac{1}{2}}. \tag{39}$$

We note that $H_\nu(c,c') = H_\nu(c',c)$.

We are now in a position to apply the standard methods of numerical analysis for the eigenvalue $\rho(\nu)$. Thus, we approximate the integral term on the right-hand side of (37) by a Gauss quadrature, namely

$$\int_0^\infty dc' H_\nu(c,c')\, \Psi_\nu(c') \simeq 3 \sum_{i=1}^{N} w_i H_\nu(c,c_i)\, \Psi_\nu(c_i), \tag{40}$$

where the Gaussian interval $(-1,1)$ has been transformed to the interval $(0, c_T)$, and w_i and c_i are the corresponding Gaussian weights and abscissae, respectively. The infinite interval in c has, for practical reasons, been reduced to the finite range $(0, c_T)$. We find that $c_T = 6$ is sufficiently high to guarantee an acceptable accuracy in the eigenvalue.

If we now choose values of $c = c_j$, and define

$$B_i = \left\{\frac{w_i c_i\, e^{-c_i^2}}{B_{11}(c_i,\nu)}\right\}^{\frac{1}{2}} \Psi_\nu(c_i), \tag{41}$$

and
$$H_{ij} = 3\{w_i w_j c_i c_j B_{11}(c_i,\nu)\, B_{11}(c_j,\nu) \exp(-c_i^2 - c_j^2)\}^{\frac{1}{2}} K(c_j, c_i), \tag{42}$$

then we can write the eigenvalue problem (37) as

$$\rho B_j = \sum_{i=1}^{N} H_{ij} B_i \quad (j = 1, \ldots, N), \tag{43}$$

or, in matrix notation, as

$$(\mathsf{H} - \rho \mathsf{I})\, \mathsf{B} = 0. \tag{44}$$

The method of Householder is now used to transform the matrix H to tri-diagonal, symmetric form. Then, with the methods of Sturm sequences and bisection, we can find as many eigenvalues as are required.

In practice, we have employed 32, 48 and 64 point Gauss quadrature, and a Richardson extrapolation was then applied to give the final result for $\rho(\nu)$. The results for $\rho(\nu)$ are shown in figure 1. Only two eigenvalues are shown $\rho_0(\nu)$ and $\rho_1(\nu)$ since these contain all the relevant information. In view of the fact that the kernel $H_\nu(c, c')$ is square integrable, any other ρ's are less than ρ_1. Indeed, it can be shown by functional analytic methods that

$$\int_0^\infty dc \int_0^\infty dc' \, |H_\nu(c, c')|^2 > \rho_0 > \rho_1 > \ldots > 0$$

(Riesz & Sz-Nagy 1955). $\rho_0(\nu)$ takes the value unity at $\nu = 0$ and increases monotonically to the value 1·218 at $\nu = 1$. It corresponds therefore to the

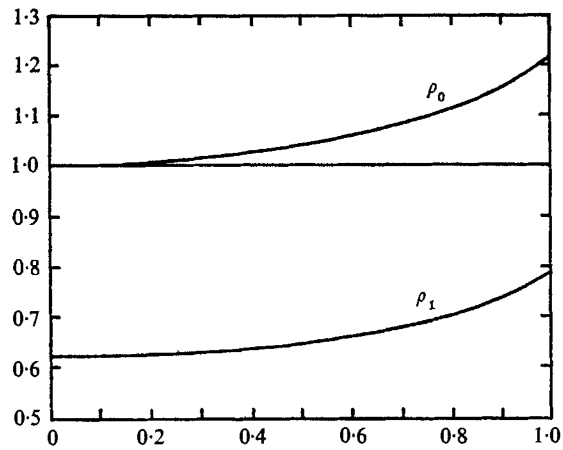

FIGURE 1. Variation of $\rho(\nu)$ for the velocity problem.

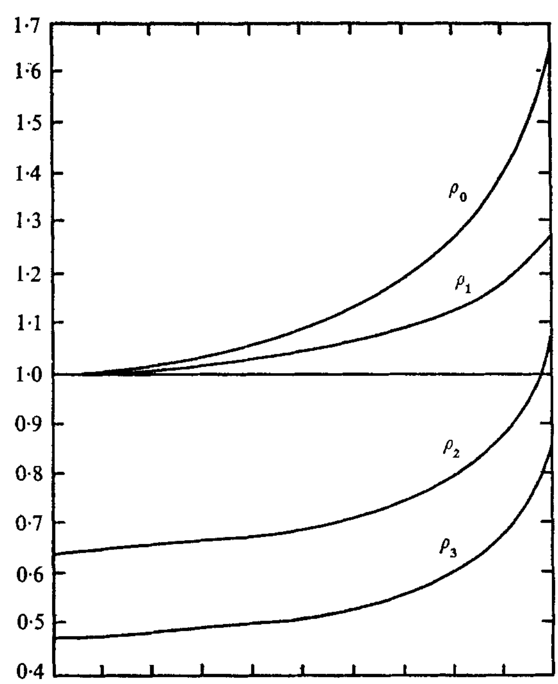

FIGURE 2. Variation of $\rho(\kappa)$ for the temperature problem.

conservation of momentum condition discussed earlier. $\rho_1(\nu)$ starts at 0·62 at $\nu = 0$ and rises monotonically to the value 0·790 at $\nu = 1$. Since $\rho_1(\nu) < 1$ for all ν, we may conclude that (36) has no non-zero eigenvalues in the range $(-1, 1)$.

Passing now to the equation of $\phi_0(c)$, and calculating $\rho(\kappa)$ in the manner described above, we arrive at figure 2. ρ_0 and ρ_1 are both equal to unity at $\nu = 0$ and they correspond to the conservation of mass and energy in the Chapman–Enskog solution. ρ_2 cuts the line $\rho = 1$ at $\kappa_0 = 0{\cdot}975$. $\rho_3 < 1$ for all κ. We may conclude therefore that there are two non-zero discrete eigenvalues in the range $(-1, 1)$ situated at $\kappa_0 = \pm 0{\cdot}975$.

As a confirmatory check on our results we have also calculated $\rho(\nu)$ for the relaxation of a neutron gas in a proton gas moderator (Williams 1966*a*, *b*, 1968). The equation to be solved is identical to (33) but with $K_0(c, c')$ given by (A 3) in the appendix. $\rho_0(\nu)$ and $\rho_1(\nu)$ were obtained. $\rho_0(\nu)$ is equal to unity at $\nu = 0$ and rises to 1·335 at $\nu = 1$. $\rho_1(\nu)$ is equal to 0·47 at $\nu = 0$ and rises monotonically to the value 0·740 at $\nu = 1$, thus indicating that there are no non-zero discrete eigenvalues in the range $(-1, 1)$. This same problem has also been solved by a WKB method (Williams 1968), from which we also obtained $\rho_1(1) = 0{\cdot}740$. This agreement gives us confidence in our calculations of the gas problems discussed above.

5. General form of solution

In view of the allowed values of ν and κ we can now write down the general form of the solution for problems involving velocity and temperature variations. In velocity problems, for example, we can write for $g(c, \mu, x)$:

$$g(c,\mu,x) = g_{\text{asy}}(c,\mu,x) + \int_1^\infty A(\nu)\, g_\nu(c,\mu)\, e^{-\nu x}\, d\nu + \int_1^\infty B(\nu)\, g_{-\nu}(c,\mu)\, e^{\nu x}\, d\nu, \quad (45)$$

where $g_{\text{asy}}(c, \mu, x)$ is the asymptotic, Chapman–Enskog solution, and the integral terms represent the contribution from the singular eigenfunctions. The unknown constants in g_{asy}, and $A(\nu)$ and $B(\nu)$, are obtained from the symmetry and boundary conditions of the problem and the solution of certain singular integral equations (Cercignani 1962). From this solution we see that the flow velocity $q(x)$ can be written in the form

$$q(x) = q_{\text{asy}}(x) + \int_1^\infty \{\alpha(\nu)\, e^{-\nu x} + \beta(\nu)\, e^{\nu x}\}\, d\nu. \quad (46)$$

If we concentrate on Kramers' problem (Williams 1969), then clearly $\beta(\nu) = 0$, and the Knudsen-layer effect decreases according to the integral term

$$\int_1^\infty \alpha(\nu)\, e^{-\nu x}\, d\nu.$$

Thus the asymptotic flow $q_{\text{asy}}(x)$ should be fully established after about two maximum mean free paths from the boundary.

In the case of temperature variations, the general form of $\phi(c, \mu, x)$ becomes

$$\phi(c,\mu,x) = \phi_{\text{asy}}(c,\mu,x) + A_0 g_{\kappa_0}(c,\mu)\, e^{-\kappa_0 x} + A_1 g_{-\kappa_0}(c,\mu)\, e^{\kappa_0 x} + \int_1^\infty A(\kappa)\, g_\kappa(c,\mu)\, e^{-\kappa x}\, d\kappa + \int_1^\infty B(\kappa)\, g_{-\kappa}(c,\mu)\, e^{\kappa x}\, d\kappa. \quad (47)$$

Here we have the additional exponential terms $\exp(\pm\kappa_0 x)$ due to the discrete eigenvalues at $\kappa_0 = \pm 0{\cdot}975$. For a temperature slip problem in a half-space, (47) would lead to the following form for the temperature distribution:

$$T(x) = T_{\text{asy}}(x) + \alpha_0 e^{-\kappa_0 x} + \int_1^\infty \alpha(\kappa)\, e^{-\kappa x}\, d\kappa, \tag{48}$$

$T_{\text{asy}}(x)$ being the Chapman–Enskog value.

Since the exponential term decays, in general, less rapidly than the integral one, we can infer that temperature (and also density) perturbations extend out further from the surface than velocity ones, i.e. the thermal Knudsen layer is 'thicker' than the velocity Knudsen layer. In practice, however, because of the proximity of κ_0 to unity, it is unlikely that this difference will be particularly marked.

6. Summary and conclusions

By studying the existence of elementary solutions of the Boltzmann equation, we have been able to build up a picture of the structure of the thermal and velocity Knudsen layers. Before any definitive conclusions can be drawn, however, it is clear that the effect of the assumptions involved in our calculations must be assessed. The most restrictive of these assumptions is the truncation of the Legendre expansion coefficients of the scattering kernel. We cannot yet state directly how this will effect the results; but a similar approximation is often made in neutron transport theory problems, where, by comparison with more accurate calculations, it does not seem to effect the general conclusions to any great extent. Physically, the truncation corresponds to a neglect of the correlation between energy loss and angle of scattering during a collision; thus it tends to underestimate the forward bias of scattering in the laboratory system of co-ordinates. In view of the fact that we can interpret κ_0 as an inverse relaxation length governing the rate at which the perturbed distribution at the wall relaxes into the asymptotic state, it seems reasonable to expect that neglect of the forward scattering bias will tend to overestimate κ_0. Thus, inclusion of additional terms in the scattering kernel would increase the importance of the discrete eigenvalue. It might even be sufficient to produce a discrete eigenvalue in the velocity problem; but this is unlikely, since $\rho_1(1)$ in that case is substantially less than unity.

Extension of the present method to other scattering models would be interesting. However, it should be noted that most of the molecular scattering models in current use (for example, the inverse power law models) have an infinite mean free path. This, of course, is spurious, as a quantum mechanical calculation will show; but it has serious mathematical consequences, and cannot be incorporated into the method described in this paper.

A number of synthetic scattering kernels are in use, the modified BGK model of Cercignani being typical. This model, however, quite clearly has no discrete, non-zero eigenvalues, and solutions of the Boltzmann equation obtained from its use always have the Knudsen layer in the integral form. It is possible that the hierarchy of models proposed by Loyalka & Ferziger (1967) will contain discrete eigenvalues, but, as yet, no numerical work has been performed.

Finally, it is worth noting that the space-eigenvalue problems discussed above have their analogies in the time domain (Grad 1963; Kuščer & Williams 1967; Rahman & Sundaresan 1968), and also in the space–time domain (Wang-Chang & Uhlenbeck 1952; Sirovich & Thurber 1963; Grad 1966; Buckner & Ferziger 1966).

Appendix

Scattering kernel for the hard-sphere gas

$$cc'K_0(c,c') = 4e^{c'^2}\,\mathrm{erf}\,(c') - \frac{4}{3\sqrt{\pi}}c'^3 - \frac{4}{\sqrt{\pi}}c^2c' \quad (c' < c), \tag{A 1}$$

$$c^2c'^2K_1(c,c') = \frac{8c'}{\sqrt{\pi}} + 4(c'^2-1)\,e^{c'^2}\,\mathrm{erf}\,(c') + \frac{4}{3\sqrt{\pi}}c'^3c^2 - \frac{4}{15\sqrt{\pi}}c'^5 \quad (c' < c). \tag{A 2}$$

For $c' > c$, interchange c' and c.

Scattering kernel for the neutron–proton gas

$$\begin{aligned} cc'K_0(c,c') &= 2e^{c'^2}\,\mathrm{erf}\,(c') \quad (c' < c), \\ &= 2e^{c^2}\,\mathrm{erf}\,(c) \quad (c' > c). \end{aligned} \tag{A 3}$$

REFERENCES

Buckner, J. K. & Ferziger, J. H. 1966 *Phys. Fluids*, **9**, 2315.

Cercignani, C. 1962 *Ann. Phys.* **20**, 219.

Chapman, S. 1917 *Phil. Trans.* A **217**, 115.

Enskog, D. 1911 *Phys. Z.* **12**, 533.

Grad, H. 1963 *Phys. Fluids*, **6**, 147.

Grad, H. 1964 *Proc. Int. Seminar on Transport Properties in Gases.* Brown University.

Grad, H. 1966 *J. SIAM, Appl. Math.* **14**, 935.

Koppel, J. 1963 *Nucl. Sci. & Engng*, **16**, 101.

Kramers, H. A. 1949 *Nuovo Cimento* suppl. **6**, 887.

Kuščer, I. & Williams, M. M. R. 1967 *Phys. Fluids*, **10**, 1922.

Loyalka, S. K. & Ferziger, J. H. 1967 *Phys. Fluids*, **10**, 1833.

Loyalka, S. K. & Ferziger, J. H. 1968 *Phys. Fluids*, **11**, 1668.

Mott-Smith, H. M. 1954 *Lincoln Laboratory, M.I.T., Rep.* V-2.

Rahman, M. & Sundaresan, M. K. 1968 *Can. J. Phys.* **21**, 2463.

Riesz, F. & Sz-Nagy, B. 1955 *Functional Analysis.* New York: Ungar.

Sirovich, L. & Thurber, J. K. 1963 *3rd Int. Symp. Rarefied Gas Dynamics*, **1**, 159.

Wang-Chang, C. S. & Uhlenbeck, G. E. 1952 *Engng Res. Inst., University of Michigan, Rep.* N 6 onr-23222, Project M 999.

Wang-Chang, C. S. & Uhlenbeck, G. E. 1953 *Engng Res. Inst., University of Michigan, Rep.* N 6 onr-23222, Project M 999.

Wang-Chang, C. S. & Uhlenbeck, G. E. 1954 *Engng Res. Inst., University of Michigan, Rep.* N 6 onr-23222.

Wang-Chang, C. S. & Uhlenbeck, G. E. 1956 *Engng Res. Inst., University of Michigan, Rep.* Non-r-122 4–15.

Welander, P. 1954 *Ark. Fys.* **7**, 507.

Williams, M. M. R. 1966*a* *The Slowing Down and Thermalization of Neutrons*. Amsterdam: North-Holland.

Williams, M. M. R. 1966*b* *Nucl. Sci. & Engng*, **26**, 262.

Williams, M. M. R. 1968 *Nucl. Sci. & Engng*, **33**, 262.

Williams, M. M. R. 1969 *J. Fluid Mech.* **36**, 145.

Wood, J. 1966 *J. Nucl. Energy*, **20**, 219.